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THE α-INVARIANT ON CERTAIN SURFACES WITH SYMMETRY GROUPS

JIAN SONG

ABSTRACT. The global holomorphic α-invariant introduced by Tian is closely related to the existence of Kähler-Einstein metrics. We apply the result of Tian, Yau and Zelditch on polarized Kähler metrics to approximate plurisubharmonic functions and compute the α-invariant on $CP^2\#n\overline{CP^2}$ for $n = 1, 2, 3$.

1. INTRODUCTION

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian [7], Tian and Yau [6] is closely related to the existence of Kähler-Einstein metrics. In his solution of the Calabi conjecture, Yau [12] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with negative or zero first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there are known obstructions such as the Futaki invariant. For a compact Kähler manifold M with positive frist Chern class, Tian [7] proved that M admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$ [9]. Nevertheless, it would also be interesting to find the estimate of the α invariant for $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$. In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman potential on polarized Kähler manifolds to approximate plurisubharmonic functions and compute the α-invariant of $CP^2\#n\overline{CP^2}$ for $n = 1, 2, 3$. In the case of $CP^2\#2\overline{CP^2}$, it gives an improvement of Abdesselem's result [1]. More precisely, we shall show that:

Theorem 1. $\alpha_G(CP^2\#2\overline{CP^2}) = \frac{1}{3}$.

We will give the definitions of the automorphism group G and the α_G-invariant in Section 3.

Let (M, ω) be a compact Kähler manifold, where $\omega=\sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j$. We will also prove Tian's conjecture on the generalized Moser-Trudinger inequality in the special case where $\alpha_G(M) > \frac{n}{n+1}$, for $n = \dim M$. Let

$$P(M,\omega) = \left\{\phi \mid \omega_\phi = \omega + \sqrt{-1}\partial\overline{\partial}\phi > 0,\ \sup_M \phi = 0\right\}.$$

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Let F_ω and J_ω be the functionals defined on $P(M,\omega)$ by

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - \log(\frac{1}{V}\int_M e^{h_\omega - \phi}\omega^n),$$

$$J_\omega(\phi) = \frac{\sqrt{-1}}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \partial\phi\wedge\overline{\partial}\phi\wedge\omega^i\wedge\omega_\phi^{n-i-1}.$$

Assume (M,ω_{KE}) is a Kähler-Einstein manifold with positive first Chern class and $Ric(\omega_{KE}) = \omega_{KE}$. Then for any $\phi\in P(M,\omega_{KE})$, Ding and Tian [2] proved the following inequality of Moser-Trudinger type:

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \le Ce^{J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n}.$$

Tian [10] also conjectured that $\frac{1}{V}\int_M e^{-\phi}\omega^n \le Ce^{(1-\delta)J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n}$ for $\delta > 0$ sufficiently small, if ϕ is perpendicular to Λ_1, the space of eigenfunctions of ω_{KE} with eigenvalue one.

We shall prove:

Theorem 2. *Let (M,ω) be a Kähler manifold with positive first Chern class. Assume that $\alpha(M) > \frac{n}{n+1}$, so that M admits a Kähler-Einstein metric ω_{KE}, and there exist constants $\delta = \delta(n,\alpha(M))$ and $C = C(n,\lambda_2(\omega_{KE}) - 1,\alpha(M))$ such that for any $\phi\in P(M,\omega_{KE})$ which satisfies $\phi\perp\Lambda_1$, we have*

$$F_{\omega_{KE}}(\phi) \ge \delta J_{\omega_{KE}}(\phi) - C.$$

Here $\lambda_2(\omega_{KE})$ is the least eigenvalue of ω_{KE} which is bigger than 1.

2. Holomorphic approximation of plurisubharmonic functions

In this section, we will employ the technique in [8, 13] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [13].

Theorem 2.1. *Let M be a compact complex manifold of dimension n and let $(L,h)\to M$ be a positive Hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m\in N$, h induces a Hermitian metric h_m on L^m. Let $\{S_0^m, S_1^m, ..., S_{d_{m-1}}^m\}$ be any orthonormal basis of $H^0(M,L^m)$, $d_m = \dim H^0(M,L^m)$, with respect to the inner product:*

$$(S_1,S_2)_{h_m} = \int_M h_m(S_1(x),S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!}\omega_g^n$ is the volume form of g. Then there is a complete asymptotic expansion

$$\sum_{i=0}^{d_m-1}||S_i^m(x)||_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + ...$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any k,

$$\Big|\Big|\sum_{i=0}^{d_m-1}||S_i^m(x)||_{h_m}^2 - \sum_{j<R}a_j(x)m^{n-j}\Big|\Big|_{C^k} \le C_{R,k}m^{n-R}$$

where $C_{R,k}$ depends on R,k and the manifold M.

Let

$$\begin{aligned}\tilde{\omega}_g &= \omega_g + \sqrt{-1}\partial\overline{\partial}\phi > 0,\\ \widetilde{h} &= he^{-\phi}.\end{aligned}$$

Let $\widetilde{h}_m$ be the induced Hermitian metric of $\widetilde{h}$ on L^m, and let $\{\widetilde{S}_0^m, \widetilde{S}_{1,\ldots,}^m \widetilde{S}_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, where $d_m = \dim H^0(M, L^m)$, with respect to the inner product

$$(S_1, S_2)_{\widetilde{h}_m} = \int_M \widetilde{h}_m(S_1(x), S_2(x))dV_{\widetilde{g}}\ .$$

By Theorem 2.1, we have

$$\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m} = \left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{h_m}\right) e^{-m\phi}.$$

Thus

$$\phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right) = -\frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right).$$

As $m \to +\infty$, we obtain for any positive integer R

$$\begin{aligned}&\frac{1}{m}\log\left(\sum_{j<R}\widetilde{a}_j(x)m^{n-j}\right)\\ =\ & \frac{1}{m}\log m^n(\sum_{j<R}\widetilde{a}_j(x)m^{-j})\\ =\ & \frac{n}{m}\log m + \frac{1}{m}\log(1+O(\frac{1}{m})) \to 0.\end{aligned}$$

Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

Corollary 2.1.

$$\left\|\phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right)\right\|_{C^k} \to 0,\ as\ m \to +\infty.$$

In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of L^m.

3. Proof of Theorem 1

Let M be the blow-up of CP^2 at two points and π be its natural projection. Without loss of generality, we may assume the two points are $p_1 = [0, 1, 0]$ and $p_2 = [0, 0, 1]$. Then M is a subvariety of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0X_1 = Z_1X_0,\quad Z_0Y_2 = Z_2Y_0,$$

where Z_i, X_j, Y_k are the homogeneous coordinates on CP^2, CP^1 and CP^1, respectively.

Let G be the automorphism group acting on $CP^2 \times CP^1 \times CP^1$ generated by θ_j and permutations τ $(0 \le j \le 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_je^{i\theta}, Z_2] \times [X_0, X_1] \times [Y_0, Y_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_2, Z_1] \times [Y_0, Y_2] \times [X_0, X_1] \ .$$

Let π_0, π_1, π_2 be the projection from $CP^2 \times CP^1 \times CP^1$ onto CP^2, CP^1 and CP^1. Respectively define ω by

$$\begin{aligned}\omega &= \pi_0^*\omega_0 + \pi_1^*\omega_1 + \pi_2^*\omega_2\\ &= \sqrt{-1}\partial\overline{\partial}\log(|Z_0|^2 + |Z_1|^2 + |Z_2|^2) + \sqrt{-1}\partial\overline{\partial}\log(|X_0|^2 + |X_1|^2)\\ &\quad + \sqrt{-1}\partial\overline{\partial}\log(|Y_0|^2 + |Y_2|^2),\end{aligned}$$

where ω_0, ω_1, ω_2 are the Fubini-Study metrics in CP^2, CP^1 and CP^1. By explicit calculation, it can be shown that the cohomological class of $\omega|_M$ is in the first Chern class of M (see [1]).

Consider the divisor

$$\{[0, Z_1, Z_2] \times CP^1 \times CP^1\} + \{CP^2 \times [1, 0] \times CP^1\} + \{CP^2 \times CP^1 \times [1, 0]\}$$

which defines a line bundle (L, h) on $CP^2 \times CP^1 \times CP^1$. The hermitian metric h is defined by

$$h = \frac{1}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|X_0|^2 + |X_1|^2)(|Y_0|^2 + |Y_2|^2)};$$

then $(L, h)|_M \to M$ defines the anticanonical line bundle on M whose curvature form $-\sqrt{-1}\partial\overline{\partial}\log h$ gives the first Chern class on M.

Since $M\backslash\{\pi^{-1}\{p_1\} \cup \pi^{-1}\{p_2\}\}$ is isomorphic to $CP^2\backslash\{p_1, p_2\}$, if we choose the inhomogeneous coordinates $(z_1, z_2) = [1, z_1, z_2]$ on CP^2, the Kähler metric

$$\omega_{g_0} = \sqrt{-1}\partial\overline{\partial}\log(1 + |z_1|^2 + |z_2|^2) + \sqrt{-1}\partial\overline{\partial}\log(1 + |z_1|^2) + \sqrt{-1}\partial\overline{\partial}\log(1 + |z_2|^2)$$

can be extended to a Kähler metric g_0 on M which belongs to $c_1(M)$. If we take different inhomogeneous coordinates $(w_0, w_1) = [w_0, w_1, 1]$, the corresponding Kähler metric is

$$\omega_{g_1} = \sqrt{-1}\partial\overline{\partial}\log(1+|w_0|^2+|w_1|^2)+\sqrt{-1}\partial\overline{\partial}\log(1+|w_0|^2)+\sqrt{-1}\partial\overline{\partial}\log(|w_0|^2+|w_1|^2)$$

and we have

$$\begin{aligned}\det g_0 &= \frac{1}{(1 + |z_1|^2 + |z_2|^2)^3} + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_1|^2)}\\ &\quad + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_2|^2)} + \frac{1}{(1 + |z_1|^2)^2(1 + |z_2|^2)^2},\\ \det g_1 &= \frac{1}{(1 + |w_0|^2 + |w_1|^2)^3} + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(|w_0|^2 + |w_1|^2)}\\ &\quad + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(1 + |w_0|^2)} + \frac{|w_0|^2}{(1 + |w_0|^2)^2(|w_0|^2 + |w_1|^2)^2}.\end{aligned}$$

Consider the line bundle $(L^N, h_N) \to CP^2 \times CP^1 \times CP^1$. Then

$$\dim H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) = \frac{(N + 1)^3(N + 2)}{2}$$

and $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ is an orthogonal basis for $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N))$.

Let M_1 be the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0X_1 = Z_1X_0,$$

and M_2 the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0 Y_2 = Z_2 Y_0.$$

Then $M = M_1 \cap M_2$.

In view of the short exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{O}(L^N - [M_1]) \rightarrow \mathcal{O}(L^N) \rightarrow \mathcal{O}(L^N|_{M_1}) \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}(L^N|_{M_1} - [M]) \rightarrow \mathcal{O}(L^N|_{M_1}) \rightarrow \mathcal{O}(L^N|_M) \rightarrow 0 \end{aligned}$$

we can choose N sufficiently large so that

$$H^1(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N - [M_1])) = H^1(M_1, \mathcal{O}(L^N|_{M_1} - [M])) = 0.$$

Then $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \rightarrow H^0(M_1, \mathcal{O}(L^N|_{M_1})) \rightarrow 0$,

$$H^0(M_1, \mathcal{O}(L^N|_{M_1})) \rightarrow H^0(M, \mathcal{O}(L^N|_M)) \rightarrow 0$$

and thus

$$H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \rightarrow H^0(M, \mathcal{O}(L^N|_M)) \rightarrow 0.$$

Also we have $Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_1}|_M = Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}$ and

$$\begin{aligned} &||Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}||^2_{h_N} \\ &\qquad = \frac{\left| Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_0^{j_0} Z_1^{j_1} Z_0^{k_0} Z_2^{k_2} \right|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(Z_0|^2 + |Z_1|^2)(Z_0|^2 + |Z_2|^2))^N} \end{aligned}$$

on $CP^2 \backslash \{p_1, p_2\}$. Therefore, $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}|_M\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ contains an orthogonal basis for $H^0(M, \mathcal{O}(L^N|_M))$ with respect to h^N and the G-invariant Kähler metric g on M.

By Corollary 2.1, for any φ in $P_G(M, \omega_g)$, we have on $CP^2 \backslash \{p_1, p_2\}$,

$$\begin{aligned} &\varphi([Z_0, Z_1, Z_2]) \\ = &\lim_{N\to\infty} \frac{1}{N} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |a^{(N)}_{(\varphi)i_0 i_1 i_2 j_0 j_1 k_0 k_2} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N} \end{aligned}$$

for some coefficients $a^{(N)}_{(\varphi)i_0 i_1 i_2 j_0 j_1 k_0 k_2}$ satisfying $a^{(N)}_{(\varphi)i_0 i_1 i_2 j_0 j_1 k_0 k_2} = a^{(N)}_{(\varphi)i_0 i_2 i_1 k_0 k_2 j_0 j_1}$ due to the group action by G.

Lemma 3.1. *Using the notations above we have*

$$\frac{1}{n} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^n} \leq 4$$

for any positive integer n.

Proof. On the patch $U_0 = \{Z_0 \neq 0\}$, let $z_1 = \frac{Z_1}{Z_0}$ and $z_2 = \frac{Z_2}{Z_0}$,

$$\begin{aligned}
&\frac{1}{n}\log\frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n}\\
\leq\ &\frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}\frac{|z_1^{i_1+j_1}z_2^{i_2+k_2}|^2}{(1+|z_1|^2+|z_2|^2)^n(1+|z_1|^2)^n(1+|z_2|^2)^n}\right)\\
\leq\ &\frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}\frac{|z_1^{i_1+j_1}z_2^{i_2+k_2}|^2}{1+|z_1^{i_1+j_1}z_2^{i_2+k_2}|^2}\right)\\
\leq\ &\frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}1\right)\\
=\ &\frac{1}{n}\log\frac{(n+1)^3(n+2)}{2}\leq 4.
\end{aligned}$$

This inequality also holds on the patch $U_1 = \{Z_1 \neq 0\}$ by continuity, and so the lemma is proved. □

Lemma 3.2. *There exists $\varepsilon > 0$ such that for any $\varphi \in P_G(M, \omega_g)$ and N, there exist $n > N$, $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ with $i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n$, and $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} > \varepsilon$.*

Proof. Otherwise, for any $\varepsilon > 0$, there exist φ and N, such that for any $n > N$ and any $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ satisfying $i_0+i_1+i_2=j_0+j_1=k_0+k_2=n$, we have $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} < \varepsilon$. By choosing n large enough and with the lemma above, we have

$$\begin{aligned}
&\varphi([Z_0, Z_1, Z_2])\\
&\leq \frac{1}{n}\log\frac{\max|a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2}|^2\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n}+\varepsilon\\
&\leq \frac{1}{n}\log\frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n}+2\log\varepsilon+\varepsilon\\
&\leq \log\varepsilon + 4.
\end{aligned}$$

Since ε could be arbitrarily small, the above inequality would imply that $\varphi \to -\infty$ uniformly, which contradicts the fact that $\sup_M \varphi = 0$. □

Proof of Theorem 1. We use notations as above; since $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} > \varepsilon$, we have

$$\begin{aligned}
&\varphi([Z_0, Z_1, Z_2])\\
=\ &\lim_{N\to\infty}\frac{1}{N}\log\frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}|a^{(N)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2}Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N}\\
\geq\ &\frac{1}{N}\log\frac{|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2+|Z_0^{i_0+j_0+k_0}Z_1^{i_2+k_2}Z_2^{i_1+j_1}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N}+\log\epsilon
\end{aligned}$$

$$\geq \frac{1}{N}\log\frac{|{Z_0}^m Z_1^{\frac{3}{2}N-\frac{m}{2}} Z_2^{\frac{3}{2}N-\frac{m}{2}}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N}+\log\epsilon$$
$$\geq \log\frac{|{Z_0}^{\frac{2m}{N}} Z_1^{3-\frac{m}{N}} Z_2^{3-\frac{m}{N}}|^2}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}+\log\epsilon,$$

where $i_0+j_0+k_0=m, i_1+j_1+i_2+k_2=3N-m$.

On the patch $U_0=\{Z_0\neq 0\}$,

$$\begin{aligned}
&\int_{U_0\cap\{0<|z_1|,|z_2|<1\}} e^{-\alpha\varphi}\omega_{g_0}^2\\
\leq\ & C_1\int_{0<|z_1|,|z_2|<1} e^{-\alpha\log\frac{|Z_0|^{\frac{2m}{N}}|Z_1|^{3-\frac{m}{N}}|Z_2|^{3-\frac{m}{N}}}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}}\omega_{g_0}^2\\
=\ & C_1\int_{0<|z_1|,|z_2|<1}\frac{(|Z_0|^2+|Z_1|^2+|Z_2|^2)^\alpha(|Z_0|^2+|Z_1|^2)^\alpha(|Z_0|^2+|Z_2|^2)^\alpha}{|Z_0|^{\frac{2\alpha m}{N}}|Z_1|^{3\alpha-\frac{\alpha m}{N}}|Z_2|^{3\alpha-\frac{\alpha m}{N}}}\omega_{g_0}^2\\
\leq\ & C_2\int_{0<|z_1|,|z_2|<1}\frac{(1+|z_1|^2+|z_2|^2)^\alpha(1+|z_1|^2)^\alpha(|1+|z_2|^2)^\alpha}{|z_1|^{3\alpha-\frac{\alpha m}{N}}|z_2|^{3\alpha-\frac{\alpha m}{N}}}dz_1\wedge d\overline{z}_1\\
&\qquad\wedge dz_2\wedge d\overline{z}_2\\
\leq\ & C_3\int_{0<|z_1|,|z_2|<1}\frac{1}{|z_1|^{3\alpha-\frac{\alpha m}{N}}|z_2|^{3\alpha-\frac{\alpha m}{N}}}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2\\
\leq\ & C_3\int_{0<|z_1|,|z_2|<1}\frac{1}{|z_1|^{3\alpha}|z_2|^{3\alpha}}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2,
\end{aligned}$$

where C_1, C_2 and C_3 are constants depending only on α and ϵ.

On the patch $U_2=\{Z_2\neq 0\}$,

$$\begin{aligned}
&\int_{U_1\cap\{0<|w_0|,|w_1|\leq 1\}} e^{-\alpha\varphi}\omega_{g_1}^2\\
\leq\ & C_4\int_{0<|w_0|,|w_1|\leq 1} e^{-\alpha\log\frac{|Z_0|^{\frac{2m}{N}}|Z_1|^{3-\frac{m}{N}}|Z_2|^{3-\frac{m}{N}}}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}}\omega_{g_1}^2\\
=\ & C_4\int_{0<|w_0|,|w_1|\leq 1}\frac{(1+|w_0|^2+|w_1|^2)^\alpha(1+|w_0|^2)^\alpha(|w_0|^2+|w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}}\omega_{g_1}^2\\
\leq\ & C_5\int_{0<|w_0|,|w_1|\leq 1}\frac{(1+|w_0|^2+|w_1|^2)^\alpha(1+|w_0|^2)^\alpha(|w_0|^2+|w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}(|w_0|^2+|w_1|^2)}dw_0\wedge d\overline{w}_0\\
&\qquad\wedge dw_1\wedge d\overline{w}_1\\
\leq\ & C_6\int_{0<|w_0|,|w_1|\leq 1}\frac{1}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}(|w_0|^2+|w_1|^2)^{1-\alpha}}dw_0\wedge d\overline{w}_0\wedge dw_1\wedge d\overline{w}_1\\
\leq\ & C_6\int_{t=0}^1\int_{s=0}^1\frac{1}{s^{\frac{\alpha m}{N}}t^{\frac{3}{2}\alpha-\frac{\alpha m}{2N}}(s+t)^{1-\alpha}}dsdt\\
\leq\ & C_6\int_{s=0}^1\frac{1}{s^{\frac{\alpha m}{N}}t^{\frac{3}{2}\alpha-\frac{\alpha m}{2N}}s^{(1-\alpha)p}t^{(1-\alpha)q}}dsdt,
\end{aligned}$$

where $p+q=1$ and C_4, C_5, C_6 are constants depending only on α and ϵ.

Case 1: If $1 \le \frac{m}{N} \le 3$, we can choose $\alpha < \min(\frac{2}{3}, \frac{1-p}{3-p})$ so that

$$\begin{aligned} \frac{\alpha m}{N} + (1-\alpha)p < 1, \\ 3\alpha - 1 < 1, \\ \frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1-\alpha)q < 1. \end{aligned}$$

Case 2: If $0 < \frac{m}{N} < 1$, we can choose $\alpha < \min(\frac{2}{3}, \frac{1-q}{3/2-q})$ so that

$$\begin{aligned} \frac{\alpha m}{N} + (1-\alpha)p < 1, \\ 3\alpha - 1 < 1, \\ \frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1-\alpha)q < 1. \end{aligned}$$

So we could choose any $\alpha < \frac{1}{3}$, which implies that $\alpha_G(M,\omega) \ge \frac{1}{3}$.

Conversely, we choose

$$\begin{aligned} \varphi_\varepsilon &= \log\left(\frac{|Z_0|^6}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)} + \varepsilon\right) \\ &\quad - \log(1+\varepsilon) \\ &\in P_G(M,\omega). \end{aligned}$$

Then we have $\sup_M \varphi_\varepsilon = 0$ and $\varphi_\varepsilon = \log\frac{\varepsilon}{1+\varepsilon}$ on the exceptional divisors. Furthermore, we have

$$\lim_{\varepsilon\to 0}\int_M e^{-\alpha\varphi_\varepsilon}\omega^2 = \infty, \ for\ any\ \alpha > \frac{1}{3}.$$

Hence we have shown $\alpha_G(M,\omega) = \frac{1}{3}$.

We can also apply the above arguments for CP^n ($n \ge 2$), $CP^2\#1\overline{CP^2}$ and $CP^2\#3\overline{CP^2}$.

(i) Let $M = CP^n$ and let G_n be the automorphism group acting on M, generated by θ_j and permutations $\tau_{i,j}$ $(0 \le i < j \le n)$,

$$\theta_j : [Z_0, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j e^{i\theta}, ..., Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau_{i,j} : [Z_0, ..., Z_i, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j, ..., Z_i, ..., Z_n].$$

Theorem 3.1. $\alpha_{G_n}(CP^n) = 1$.

(ii) Let M be the blow-up of CP^2 at 3 points which are not collinear. Then we can assume that these 3 points are $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$. Let $G(3)$ be the automorphism group acting on M, generated by θ_j and permutations $\tau_{i,j}$ $(0 \le i < j \le 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \to [Z_0, Z_j e^{i\theta}, Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau_{i,j} : [..., Z_i, ..., Z_j, ...] \to [..., Z_j, ..., Z_i, ...].$$

Theorem 3.2. $\alpha_{G(3)}(CP^2\#3\overline{CP^2}) = 1$.

(iii) Let M be the blow-up of CP^2 at one point $[1,0,0]$ and $G(1)$ be the automorphism group acting on M, generated by θ_j and permutations τ $(0 \leq i \leq 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \to [Z_0, Z_j e^{i\theta}, Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \to [Z_0, Z_2, Z_1].$$

Theorem 3.3. $\alpha_{G(1)}(CP^2 \# 1\overline{CP^2}) = \frac{1}{2}$.

Also the proof above shows that the sequence of the holomorphic invariants $\{\alpha_{G,m}(M)\}_m$ defined by Tian [8] on CP^n $(n \geq 2)$, $CP^2 \# k\overline{CP^2}$ $(k = 1, 2, 3)$ is stationary.

4. Proof of Theorem 2

In this section, we will prove the generalized Moser-Trudinger inequality on any Kähler manifold M of dimension n whose $\alpha(M)$ is greater than $\frac{n}{n+1}$. The following theorem is due to Tian and Zhu [11].

Theorem 4.1. *Let (M, ω) be a Kähler-Einstein manifold with $Ric(\omega) = \omega$; then there exist constants $\delta = \delta(n)$ and $C = C\left(n, \lambda_2(\omega) - 1\right) \geq 0$ such that for any $\phi \in P(M, \omega)$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_\omega(\phi) \geq J_\omega(\phi)^\delta - C,$$

which is the same as

$$\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{J_\omega(\phi) - \frac{1}{V}\int_M \phi \omega^n - J_\omega(\phi)^\delta}.$$

This implies in particular the Moser-Trudinger inequality on S^2, which reads

$$\frac{1}{4\pi} \int_{S^2} e^{-\phi} \omega \leq e^{\frac{1}{8\pi}\int_{S^2} |\nabla \phi|^2 \omega - \frac{1}{4\pi}\int_{S^2} \phi}.$$

For any $\phi \in P(M, \omega)$, put $\omega' = \omega_\phi = \omega + \sqrt{-1}\partial\overline{\partial}\phi$ and $Ric(\omega) = \omega + \sqrt{-1}\partial\overline{\partial}h_\omega$. Consider the Monge-Ampère equation

$$(\omega' + \sqrt{-1}\partial\overline{\partial}\psi)^n = e^{h_\omega - t\psi}\omega'^n.$$

We will use the continuity method backwards and let ϕ_t be a smooth family which solve the above equation.

The following lemmas are well known [10], but we add the proofs for the sake of completeness.

Lemma 4.1. *$Ric(\omega_t) \geq t\omega_t$ and we have equality if and only if $t = 1$, where $\omega_t = \omega + \phi_t$ and ϕ_t solves the Monge-Ampère equation at t.*

Proof.

$$\begin{aligned} Ric(\omega_t) &= -\sqrt{-1}\partial\overline{\partial}\log\omega_t^n = -\sqrt{-1}\partial\overline{\partial}\log\frac{\omega_t^n}{\omega^n} + Ric(\omega) \\ &= -\sqrt{-1}\partial\overline{\partial}\left(h_\omega - t\phi_t\right) + \omega + \sqrt{-1}\partial\overline{\partial}h_\omega \\ &= \omega + t\phi_t = t\omega_t + (1-t)\omega \geq t\omega_t. \end{aligned}$$

□

Lemma 4.2. *For any $\phi \in P(M,\omega)$, if the Green's function of $\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\phi$ is bounded from below, we have:*

$$-\inf_M \phi \leq \frac{1}{V}\int_M (-\phi)\omega'^n + C.$$

Proof. Since $\omega + \sqrt{-1}\partial\overline{\partial}\phi = \omega'$ and $\omega' - \sqrt{-1}\partial\overline{\partial}\phi > 0$, we have $\Delta_{\omega'}\phi \leq n$, and

$$\begin{aligned} -\phi &= \frac{1}{V}\int_M (-\phi)\omega'^n + \frac{1}{V}\int_M \Delta_{\omega'}\phi(y) G_{\omega'}(x,y)\omega'^n \\ &\leq \frac{1}{V}\int_M (-\phi)\omega'^n + \frac{1}{V}\int_M n(G_{\omega'}(x,y) - \inf G_{\omega'}(x,y))\omega'^n \\ &\leq \frac{1}{V}\int_M (-\phi)\omega'^n + C. \end{aligned}$$

□

Let (M,ω) be a Kähler-Einstein manifold with $\mathrm{Ric}(\omega) = \omega$ and let $P(M,\omega,K) = \{\phi \in P(M,\omega) \mid G_{\omega+\sqrt{-1}\partial\overline{\partial}\phi}(x,y) \geq -K)$. Then we have:

Proposition 4.1. *Let (M,ω) be a Kähler-Einstein manifold with $Ric\,(\omega) = \omega$. If $\alpha(M) > \frac{n}{n+1}$, then there exist constants $\delta(n,\alpha,K)$ and $C(n,\alpha,\lambda_2(\omega)-1,K)$ such that for any $\phi \in P(M,\omega,K)$, we have*

$$F_\omega(\phi) \geq \delta J_\omega(\phi) - C.$$

Proof. Let $\omega' = \omega + \partial\overline{\partial}\phi$, where $\phi \in P(M,\omega,K)$. We have

$$\begin{aligned} \frac{1}{V}\int_M e^{-\alpha\phi}\omega^n &= \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2+\varepsilon)\phi}\omega^n \\ &\leq \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2)\phi}\omega^n e^{-\varepsilon\inf_M \phi}; \end{aligned}$$

taking $p = \frac{1}{\alpha_1}, q = \frac{1}{1-\alpha_1}$, we have

$$\begin{aligned} \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2)\phi}\omega^n &\leq \frac{1}{V}(\int_M e^{-\alpha_1 p\phi}\omega^n)^{1/p}(\int_M e^{-\alpha_2 q\phi}\omega^n)^{1/q} \\ &= \frac{1}{V}(\int_M e^{-\phi}\omega^n)^{\alpha_1}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{1-\alpha_1} \\ &\leq Ce^{\alpha_1 J_\omega(\phi) - \frac{\alpha_1}{V}\int_M \phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{1-\alpha_1}. \end{aligned}$$

By Lemma 4.2,

$$\begin{aligned} e^{-\varepsilon\inf_M \phi} &\leq e^{\frac{\varepsilon}{V}\int_M(-\phi)\omega'^n + C} \\ &= e^{\varepsilon I_\omega(\phi) - \frac{\varepsilon}{V}\int_M \phi\omega^n + C} \\ &\leq e^{\varepsilon(n+1)J_\omega(\phi) - \frac{\varepsilon}{V}\int_M \phi\omega^n + C}. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}\frac{1}{V}\int_M e^{-\phi}\omega^n &\leq (\frac{1}{V}\int_M e^{-\alpha\phi}\omega^n)^{\frac{1}{\alpha}} \\ &\leq Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{\alpha_1+\varepsilon}{\alpha V}\int_M \phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{\frac{1-\alpha_1}{\alpha}} \\ &= Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{1}{V}\int_M \phi\omega^n+\frac{\alpha_2}{V}\int_M(\phi-\sup\phi)\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n)^{\frac{1-\alpha_1}{\alpha}} \\ &\leq Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{1}{V}\int_M \phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n)^{\frac{1-\alpha_1}{\alpha}}.\end{aligned}$$

We need to determine $\alpha_1,\ \alpha_2,\ \varepsilon$ which satisfy the following conditions:

$$\begin{aligned}\alpha &= \alpha_1+\alpha_2+\varepsilon>1, \\ \alpha &> \alpha_1+(n+1)\varepsilon, \\ 1 &> \alpha_1.\end{aligned}$$

So we will choose

$$\begin{aligned}\alpha_2 &= n\varepsilon+\varepsilon', \\ \alpha_1 &= 1-\alpha_2-\varepsilon+\varepsilon''=1-(n+1)\varepsilon-\varepsilon'+\varepsilon'',\end{aligned}$$

where ε , ε' , $\varepsilon''<<1$, and $\varepsilon'=o(\varepsilon)$, $\varepsilon''=o(\varepsilon')$.

Since $\alpha(M)>\frac{n}{n+1}$, we can choose ε, ε' , ε'' small enough; then we have

$$\frac{\alpha_2}{1-\alpha_1}=\frac{n\varepsilon+\varepsilon'}{(n+1)\varepsilon+\varepsilon'-\varepsilon''}<\alpha(M)$$

and

$$\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n<Const.$$

Combined with the inequalities above, we have

$$\frac{1}{V}\int_M e^{-\phi}\omega^n\leq Ce^{(1-\delta)J_\omega(\phi)-\frac{1}{V}\int_M\phi\omega^n}.$$

This proves the lemma. □

Proof of Theorem 2. We assume ω is the Kähler-Einstein metric of M. For any $\phi\in P(M,\omega)$, put $\omega'=\omega+\sqrt{-1}\partial\overline{\partial}\phi$. Consider $(\omega'+\sqrt{-1}\partial\overline{\partial}\psi)=e^{h_{\omega'}+t\psi}$. By solving the Monge-Ampère equation backwards, we get the solutions ϕ_t, and $\phi_1=-\phi$.

For $t>\frac{1}{2}$, let $\omega_t=\omega'+\sqrt{-1}\partial\overline{\partial}\phi_t=\omega+\sqrt{-1}\partial\overline{\partial}(\phi_t-\phi_1)$; by Lemma 4.1,

$$Ric(\omega_t)\geq\frac{1}{2}\omega_t,$$

which shows that the Green function of ω_t is uniformly bounded from below. Thus by Proposition 4.1 and the calculation in [11] we have

$$\begin{aligned}F_\omega(\phi_t-\phi_1) &\geq \delta J_\omega(\phi_t-\phi_1)-C \\ &\geq C_1osc_M(\phi_t-\phi_1)-C_2,\end{aligned}$$

and consequently,

$$\begin{aligned} n(1-t)J_\omega(\phi) &= n(1-t)J_{\omega'}(\phi_1) \\ &\geq (1-t)(I_{\omega'}(\phi_1) - J_{\omega'}(\phi_1)) \\ &\geq F_{\omega'}(\phi_t) - F_{\omega'}(\phi_1) \\ &= F_\omega(\phi_t - \phi_1) \\ &\geq C_1 osc_M(\phi_t - \phi_1) - C_2. \end{aligned}$$

Thus we have

$$\begin{aligned} F_\omega(\phi) &= -F_{\omega'}(-\phi) \\ &= \int_0^1 (I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t))dt \\ &\geq (1-t)(I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t)) \\ &\geq \frac{1-t}{n} J_{\omega'}(\phi_t) \\ &\geq \frac{1-t}{n} J_{\omega'}(\phi_1) - 2(1-t)(C_1 osc_M(\phi_t - \phi_1) - C_2) \\ &\geq \frac{1-t}{n} J_\omega(\phi) - 2(1-t)^2 n C_1 J_\omega(\phi) - C_3. \end{aligned}$$

The theorem follows by choosing $(1-t) < \frac{1}{2n^2C_1}$.

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References

[1] Ben Abdesselem, A., *Équations de Monge-Ampère d'origine géométrique sur certaines variétés algébriques*, J.Funct. Anal. 149 (1997), 102–134. MR **98i:**32020

[2] Ding, W and Tian, G., *The generalized Moser-Trudinger Inequality*, Proceedings of Nankai International Conference on Nonlinear Analysis, 1993.

[3] Lu, Z., *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math. 122 (2000), no. 2, 235–273. MR **2002d:**32034

[4] Phong, D. H. and Sturm, J., *Algebraic estimates, stability of local zeta functions, and uniform estimates for distribution functions*, Ann. of Math. (2) 152 (2000), no. 1, 277–329. MR **2002f:**11180

[5] Siu, Y.T., *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. of Math. (2) 127 (1988), no. 3, 585–627. MR **89e:**58032

[6] Tian, G. and Yau, S.T., *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$*, Comm. Math. Phys. 112 (1987), no.1, 175–203. MR **88k:**32070

[7] Tian, G., *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math. 89 (1987), no. 2, 225–246. MR **88e:**53069

[8] Tian, G., *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geometry 32 (1990), 99–130. MR **91j:**32031

[9] Tian, G., *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101 (1990), no. 1, 101–172. MR **91d:**32042

[10] Tian, G., *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. 130 (1997), no. 1, 1–37. MR **99e:**53065

[11] Tian, G. and Zhu, X.H., *A nonlinear inequality of Moser-Trudinger type*, Calc. Var. Partial Differential Equations 10 (2000), no. 4, 349–354. MR **2001f:**32044

[12] Yau, S.T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math. 31 (1978), 339–411. MR **81d:**53045

[13] Zelditch, S., *Szegö Kernels and a Theorem of Tian*, IMRN1998, No. 6, 317–331. MR **99g:**32055

Department of Mathematics, Columbia University, New York, New York 10027
E-mail address: jsong@math.columbia.edu